

# ON SHARP APERTURE-WEIGHTED ESTIMATES FOR SQUARE FUNCTIONS

ANDREI K. LERNER

ABSTRACT. Let  $S_{\alpha,\psi}(f)$  be the square function defined by means of the cone in  $\mathbb{R}_+^{n+1}$  of aperture  $\alpha$ , and a standard kernel  $\psi$ . Let  $[w]_{A_p}$  denote the  $A_p$  characteristic of the weight  $w$ . We show that for any  $1 < p < \infty$  and  $\alpha \geq 1$ ,

$$\|S_{\alpha,\psi}\|_{L^p(w)} \lesssim \alpha^n [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})}.$$

For each fixed  $\alpha$  the dependence on  $[w]_{A_p}$  is sharp. Also, on all class  $A_p$  the result is sharp in  $\alpha$ . Previously this estimate was proved in the case  $\alpha = 1$  using the intrinsic square function. However, that approach does not allow to get the above estimate with sharp dependence on  $\alpha$ . Hence we give a different proof suitable for all  $\alpha \geq 1$  and avoiding the notion of the intrinsic square function.

## 1. INTRODUCTION

Let  $\psi$  be an integrable function,  $\int_{\mathbb{R}^n} \psi = 0$ , and, for some  $\varepsilon > 0$ ,

$$(1.1) \quad |\psi(x)| \leq \frac{c}{(1+|x|)^{n+\varepsilon}} \quad \text{and} \quad \int_{\mathbb{R}^n} |\psi(x+h) - \psi(x)| dx \leq c|h|^\varepsilon.$$

Let  $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$  and  $\Gamma_\alpha(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |y - x| < \alpha t\}$ . Set  $\psi_t(x) = t^{-n}\psi(x/t)$ . Define the square function  $S_{\alpha,\psi}(f)$  by

$$S_{\alpha,\psi}(f)(x) = \left( \int_{\Gamma_\alpha(x)} |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \quad (\alpha > 0).$$

We drop the subscript  $\alpha$  if  $\alpha = 1$ .

Given a weight  $w$ , define its  $A_p$  characteristic by

$$[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w dx \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} dx \right)^{p-1},$$

where the supremum is taken over all cubes  $Q \subset \mathbb{R}^n$ .

---

2010 *Mathematics Subject Classification.* 42B20, 42B25.

*Key words and phrases.* Littlewood-Paley operators, sharp weighted inequalities, sharp aperture dependence.

It was proved in [13] that for any  $1 < p < \infty$ ,

$$(1.2) \quad \|S_\psi\|_{L^p(w)} \leq c_{p,n,\psi} [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})},$$

and this estimate is sharp in terms of  $[w]_{A_p}$  (we also refer to [13] for a detailed history of closely related results).

Similarly one can show that

$$(1.3) \quad \|S_{\alpha,\psi}\|_{L^p(w)} \leq c_{p,n,\psi} \gamma(\alpha) [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})} \quad (\alpha \geq 1, 1 < p < \infty);$$

however, the sharp dependence on  $\alpha$  in this estimate cannot be determined by means of the approach from [13]. The aim of this paper is to find the sharp  $\gamma(\alpha)$  in (1.3).

Let us explain first why the method from [13] gives a rough estimate for  $\gamma(\alpha)$ . The proof in [13] was based on the intrinsic square function  $G_{\alpha,\beta}(f)$  by M. Wilson [19] defined as follows. For  $0 < \beta \leq 1$ , let  $\mathcal{C}_\beta$  be the family of functions supported in the unit ball with mean zero and such that for all  $x$  and  $x'$ ,  $|\varphi(x) - \varphi(x')| \leq |x - x'|^\beta$ . If  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  and  $(y, t) \in \mathbb{R}_+^{n+1}$ , we define  $A_\beta(f)(y, t) = \sup_{\varphi \in \mathcal{C}_\beta} |f * \varphi_t(y)|$  and

$$G_{\alpha,\beta}(f)(x) = \left( \int_{\Gamma_\alpha(x)} (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

Set  $G_{1,\beta}(f) = G_\beta(f)$ .

The intrinsic square function has several interesting features (established in [19]). First, though  $G_\beta(f)$  is defined by means of kernels with uniform compact support, it pointwise dominates  $S_\psi(f)$ . Also there is a pointwise relation between  $G_{\alpha,\beta}(f)$  with different apertures:

$$(1.4) \quad G_{\alpha,\beta}(f)(x) \leq \alpha^{(3/2)n+\beta} G_\beta(f)(x) \quad (\alpha \geq 1).$$

Notice that for the usual square functions  $S_{\alpha,\psi}(f)$  such a pointwise relation is not available.

In [13], (1.2) with  $G_\beta(f)$  instead of  $S_\psi(f)$  was obtained. Combining this with (1.4), we would obtain that one can take  $\gamma(\alpha) = \alpha^{(3/2)n+\beta}$  in (1.3) assuming that  $\psi$  is compactly supported. For non-compactly supported  $\psi$  some additional ideas from [19] can be used that lead to even worst estimate on  $\gamma(\alpha)$ . Observe also that it is not clear to us whether (1.4) can be improved.

It is easy to see that the dependence  $\gamma(\alpha) = \alpha^{(3/2)n+\beta}$  in (1.3) is far from the sharp one. For instance, it is obvious that the information on  $\beta$  should not appear in (1.3). All this indicates that the intrinsic square function approach is not suitable for our purposes in determining the sharp  $\gamma(\alpha)$ .

Suppose we seek for  $\gamma(\alpha)$  in the form  $\gamma(\alpha) = \alpha^r$ . Then a simple observation shows that  $r \geq n$  for any  $1 < p < \infty$ . Indeed, consider the Littlewood-Paley function  $g_{\mu,\psi}^*(f)$  defined by

$$g_{\mu,\psi}^*(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\mu n} |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

Using the standard estimate

$$g_{\mu,\psi}^*(f)(x) \leq S_\psi(f)(x) + \sum_{k=0}^{\infty} 2^{-k\mu n/2} S_{2^{k+1},\psi}(f)(x),$$

we obtain that (1.3) for some  $p = p_0$  and  $\gamma(\alpha) = \alpha^{r_0}$  implies

$$(1.5) \quad \|g_{\mu,\psi}^*\|_{L^{p_0}(w)} \lesssim \left( \sum_{k=0}^{\infty} 2^{-k\mu n/2} 2^{kr_0} \right) [w]_{A_{p_0}}^{\max(\frac{1}{2}, \frac{1}{p_0-1})}.$$

This means that if  $\mu > 2r_0/n$ , then  $g_{\mu,\psi}^*$  is bounded on  $L^{p_0}(w)$ ,  $w \in A_{p_0}$ . From this, by the Rubio de Francia extrapolation theorem,  $g_{\mu,\psi}^*$  is bounded on the unweighted  $L^p$  for any  $p > 1$ , whenever  $\mu > 2r_0/n$ . But it is well known [8] that  $g_{\mu,\psi}^*$  is not bounded on  $L^p$  if  $1 < \mu < 2$  and  $1 < p \leq 2/\mu$ . Hence, if  $r_0 < n$ , we would obtain a contradiction to the latter fact for  $p$  sufficiently close to 1.

Our main result shows that for any  $1 < p < \infty$  one can take the optimal power growth  $\gamma(\alpha) = \alpha^n$ .

**Theorem 1.1.** *For any  $1 < p < \infty$  and for all  $1 \leq \alpha < \infty$ ,*

$$\|S_{\alpha,\psi}\|_{L^p(w)} \leq c_{p,n,\psi} \alpha^n [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})}.$$

By (1.5), we immediately obtain the following.

**Corollary 1.2.** *Let  $\mu > 2$ . Then for any  $1 < p < \infty$ ,*

$$\|g_{\mu,\psi}^*(f)\|_{L^p(w)} \leq c_{p,n,\mu,\psi} [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})}.$$

Observe that if  $\mu = 2$ , then  $g_{2,\psi}^*$  is also bounded on  $L^p(w)$  for  $w \in A_p$  (see [17]). However, the sharp dependence on  $[w]_{A_p}$  in the corresponding  $L^p(w)$  inequality is unknown to us.

We emphasize that the growth  $\gamma(\alpha) = \alpha^n$  is best possible in the weighted  $L^p(w)$  estimate for  $w \in A_p$ . In the unweighted case a better dependence on  $\alpha$  is known, namely,  $\|S_{\alpha,\psi}\|_{L^p} \leq c_{p,n,\psi} \alpha^{\frac{n}{\min(p,2)}}$ , see [1, 18].

Some words about the proof of Theorem 1.1. As in [13], we use here the local mean oscillation decomposition. But in [13] we worked with the intrinsic square function, and due to the fact that this operator

is defined by uniform compactly supported kernels, we arrived to the operator

$$\mathcal{A}(f)(x) = \left( \sum_{j,k} (f_{\gamma Q_j^k})^2 \chi_{Q_j^k}(x) \right)^{1/2},$$

where  $Q_j^k$  is a sparse family and  $\gamma > 1$ . This operator can be handled sufficiently easy.

Here we work with the square function  $S_{\alpha,\psi}(f)$  directly, more precisely we consider its smooth variant  $\tilde{S}_{\alpha,\psi}(f)$ . Applying the local mean oscillation decomposition to  $\tilde{S}_{\alpha,\psi}(f)$ , we obtain that  $S_{\alpha,\psi}(f)$  is essentially pointwise bounded by  $\alpha^n \mathcal{B}(f)$ , where

$$\mathcal{B}(f)(x) = \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \left( \sum_{j,k} (f_{2^m Q_j^k})^2 \chi_{Q_j^k}(x) \right)^{1/2} \quad (\delta > 0).$$

Observe that this pointwise aperture estimate is interesting in its own right. In order to handle  $\mathcal{B}$ , we use a mixture of ideas from recent papers on a simple proof of the  $A_2$  conjecture [14] and sharp weighted estimates for multilinear Calderón-Zygmund operators [5]. In particular, similarly to [14], we obtain the  $X^{(2)}$ -norm boundedness of  $\mathcal{B}$  by  $\mathcal{A}$  on an arbitrary Banach function space  $X$ .

The paper is organized as follows. Next section contains some preliminary information. In Section 3, we obtain the main estimate, namely, the local mean oscillation estimate of  $\tilde{S}_{\alpha,\psi}(f)$ . The proof of Theorem 1.1 is contained in Section 4. Section 5 contains some concluding remarks concerning the sharp aperture-weighted weak type estimates for  $S_{\alpha,\psi}(f)$ .

## 2. PRELIMINARIES

**2.1. A weak type  $(1, 1)$  estimate for square functions.** It is well known that the operator  $S_{\alpha,\psi}$  is of weak type  $(1, 1)$ . However, we could not find in the literature the sharp dependence on  $\alpha$  in the corresponding inequality. Hence we give below an argument based on general square functions.

For a measurable function  $F$  on  $\mathbb{R}_+^{n+1}$  define

$$S_{\alpha}(F)(x) = \left( \int_{\Gamma_{\alpha}(x)} |F(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

**Lemma 2.1.** *For any  $\alpha \geq 1$ ,*

$$(2.1) \quad \|S_{\alpha}(F)\|_{L^{1,\infty}} \leq c_n \alpha^n \|S_1(F)\|_{L^{1,\infty}}.$$

*Proof.* We will use the following estimate which can be found in [18, p. 315]: if  $\Omega \subset \mathbb{R}^n$  is an open set and  $U = \{x \in \mathbb{R}^n : M\chi_\Omega(x) > 1/2\alpha^n\}$ , where  $M$  is the Hardy-Littlewood maximal operator, then

$$\int_{\mathbb{R}^n \setminus U} S_\alpha(F)(x)^2 dx \leq 2\alpha^n \int_{\mathbb{R}^n \setminus \Omega} S_1(F)(x)^2 dx$$

(observe that the definitions of  $S_\alpha(F)$  here and in [18] are differ by the factor  $\alpha^{n/2}$ .)

Let  $\Omega_\xi = \{x : S_1(F)(x) > \xi\}$ . Using the weak type  $(1, 1)$  of  $M$ , Chebyshev's inequality and the above estimate, we obtain

$$\begin{aligned} & |\{x \in \mathbb{R}^n : S_\alpha(F)(x) > \xi\}| \\ & \leq |U_\xi| + |\{x \in \mathbb{R}^n \setminus U_\xi : S_\alpha(F)(x) > \xi\}| \\ & \leq c_n \alpha^n |\{x : S_1(F)(x) > \xi\}| + \frac{1}{\xi^2} \int_{\mathbb{R}^n \setminus U_\xi} S_\alpha(F)(x)^2 dx \\ & \leq c_n \alpha^n |\{x : S_1(F)(x) > \xi\}| + \frac{2\alpha^n}{\xi^2} \int_{\mathbb{R}^n \setminus \Omega_\xi} S_1(F)(x)^2 dx. \end{aligned}$$

Further,

$$\int_{\mathbb{R}^n \setminus \Omega_\xi} S_1(F)(x)^2 dx \leq 2 \int_0^\xi \lambda |\{x : S_1(F)(x) > \lambda\}| d\lambda \leq 2\xi \|S_1(F)\|_{L^{1,\infty}}.$$

Combining this with the previous estimate gives

$$|\{x : S_\alpha(F)(x) > \xi\}| \leq c_n \alpha^n |\{x : S_1(F)(x) > \xi\}| + \frac{4\alpha^n}{\xi} \|S_1(F)\|_{L^{1,\infty}},$$

which proves (2.1).  $\square$

Note that the sharp strong  $L^p$  estimates related square functions of different apertures were obtained recently in [1].

By Lemma 2.1 and by the weak type  $(1, 1)$  of  $S_\psi(f)$  [9],

$$(2.2) \quad \|S_{\alpha,\psi}(f)\|_{L^{1,\infty}} \leq c_{n,\psi} \alpha^n \|f\|_{L^1}.$$

**2.2. Dyadic grids and sparse families.** Recall that the standard dyadic grid in  $\mathbb{R}^n$  consists of the cubes

$$2^{-k}([0, 1]^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.$$

Denote the standard grid by  $\mathcal{D}$ .

By a *general dyadic grid*  $\mathcal{D}$  we mean a collection of cubes with the following properties: (i) for any  $Q \in \mathcal{D}$  its sidelength  $\ell_Q$  is of the form  $2^k, k \in \mathbb{Z}$ ; (ii)  $Q \cap R \in \{Q, R, \emptyset\}$  for any  $Q, R \in \mathcal{D}$ ; (iii) the cubes of a fixed sidelength  $2^k$  form a partition of  $\mathbb{R}^n$ .

Given a cube  $Q_0$ , denote by  $\mathcal{D}(Q_0)$  the set of all dyadic cubes with respect to  $Q_0$ , that is, the cubes from  $\mathcal{D}(Q_0)$  are formed by repeated

subdivision of  $Q_0$  and each of its descendants into  $2^n$  congruent sub-cubes. Observe that if  $Q_0 \in \mathcal{D}$ , then each cube from  $\mathcal{D}(Q_0)$  will also belong to  $\mathcal{D}$ .

We will use the following proposition from [10].

**Proposition 2.2.** *There are  $2^n$  dyadic grids  $\mathcal{D}_i$  such that for any cube  $Q \subset \mathbb{R}^n$  there exists a cube  $Q_i \in \mathcal{D}_i$  such that  $Q \subset Q_i$  and  $\ell_{Q_i} \leq 6\ell_Q$ .*

We say that  $\{Q_j^k\}$  is a *sparse family* of cubes if: (i) the cubes  $Q_j^k$  are disjoint in  $j$ , with  $k$  fixed; (ii) if  $\Omega_k = \cup_j Q_j^k$ , then  $\Omega_{k+1} \subset \Omega_k$ ; (iii)  $|\Omega_{k+1} \cap Q_j^k| \leq \frac{1}{2}|Q_j^k|$ .

**2.3. A “local mean oscillation decomposition”.** The non-increasing rearrangement of a measurable function  $f$  on  $\mathbb{R}^n$  is defined by

$$f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| < \alpha\}| < t\} \quad (0 < t < \infty).$$

Given a measurable function  $f$  on  $\mathbb{R}^n$  and a cube  $Q$ , the local mean oscillation of  $f$  on  $Q$  is defined by

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^*(\lambda|Q|) \quad (0 < \lambda < 1).$$

By a median value of  $f$  over  $Q$  we mean a possibly nonunique, real number  $m_f(Q)$  such that

$$\max(|\{x \in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}|) \leq |Q|/2.$$

It is easy to see that the set of all median values of  $f$  is either one point or the closed interval. In the latter case we will assume for the definiteness that  $m_f(Q)$  is the *maximal* median value. Observe that it follows from the definitions that

$$(2.3) \quad |m_f(Q)| \leq (f\chi_Q)^*(|Q|/2).$$

Given a cube  $Q_0$ , the dyadic local sharp maximal function  $m_{\lambda; Q_0}^{\#, d} f$  is defined by

$$m_{\lambda; Q_0}^{\#, d} f(x) = \sup_{x \in Q' \in \mathcal{D}(Q_0)} \omega_\lambda(f; Q').$$

The following theorem was proved in [15] (its very similar version can be found in [12]).

**Theorem 2.3.** *Let  $f$  be a measurable function on  $\mathbb{R}^n$  and let  $Q_0$  be a fixed cube. Then there exists a (possibly empty) sparse family of cubes  $Q_j^k \in \mathcal{D}(Q_0)$  such that for a.e.  $x \in Q_0$ ,*

$$|f(x) - m_f(Q_0)| \leq 4m_{\frac{1}{2^{n+2}}; Q_0}^{\#, d} f(x) + 2 \sum_{k, j} \omega_{\frac{1}{2^{n+2}}}(f; Q_j^k) \chi_{Q_j^k}(x).$$

## 3. A KEY ESTIMATE

In this section we will obtain the main local mean oscillation estimate of  $S_{\alpha,\psi}$ . We consider a smooth version of  $S_{\alpha,\psi}$  defined as follows. Let  $\Phi$  be a Schwartz function such that

$$\chi_{B(0,1)}(x) \leq \Phi(x) \leq \chi_{B(0,2)}(x).$$

Define

$$\tilde{S}_{\alpha,\psi}(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \Phi\left(\frac{x-y}{t\alpha}\right) |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \quad (\alpha > 0).$$

It is easy to see that

$$S_{\alpha,\psi}(f)(x) \leq \tilde{S}_{\alpha,\psi}(f)(x) \leq S_{2\alpha,\psi}(f)(x).$$

Hence, by (2.2),

$$(3.1) \quad \|\tilde{S}_{\alpha,\psi}(f)\|_{L^{1,\infty}} \leq c_{n,\psi} \alpha^n \|f\|_{L^1}.$$

**Lemma 3.1.** *For any cube  $Q \subset \mathbb{R}^n$ ,*

$$(3.2) \quad \omega_\lambda(\tilde{S}_{\alpha,\psi}(f)^2; Q) \leq c_{n,\lambda,\psi} \alpha^{2n} \sum_{k=0}^{\infty} \frac{1}{2^{k\delta}} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2,$$

where  $\delta = \varepsilon$  from condition (1.1) if  $\varepsilon < 1$ , and  $\delta < 1$  if  $\varepsilon = 1$ .

*Proof.* Given a cube  $Q$ , let  $T(Q) = \{(y, t) : y \in Q, 0 < t < \ell_Q\}$ , where  $\ell_Q$  denotes the side length of  $Q$ . For  $x \in Q$  we decompose  $\tilde{S}_{\alpha,\psi}(f)(x)^2$  into the sum of

$$I_1(f)(x) = \iint_{T(2Q)} \Phi\left(\frac{x-y}{t\alpha}\right) |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}}$$

and

$$I_2(f)(x) = \iint_{\mathbb{R}_+^{n+1} \setminus T(2Q)} \Phi\left(\frac{x-y}{t\alpha}\right) |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}}.$$

Let us show first that

$$(3.3) \quad (I_1(f)\chi_Q)^*(\lambda|Q|) \leq c_{n,\lambda,\psi} \alpha^{2n} \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2.$$

Using that  $(a+b)^2 \leq 2(a^2+b^2)$ , we get

$$I_1(f)(x) \leq 2(I_1(f\chi_{4Q})(x) + I_1(f\chi_{\mathbb{R}^n \setminus 4Q})(x)).$$

Hence,

$$(3.4) \quad \begin{aligned} (I_1(f)\chi_Q)^*(\lambda|Q|) &\leq 2((I_1(f\chi_{4Q}))^*(\lambda|Q|/2) \\ &\quad + (I_1(f\chi_{\mathbb{R}^n \setminus 4Q})\chi_Q)^*(\lambda|Q|/2)). \end{aligned}$$

By (3.1),

$$(3.5) \quad \begin{aligned} (I_1(f\chi_{4Q}))^*(\lambda|Q|/2) &\leq (\tilde{S}_{\alpha,\psi}(f\chi_{4Q}))^*(\lambda|Q|/2)^2 \\ &\leq c_{n,\lambda,\psi}\alpha^{2n} \left( \frac{1}{|4Q|} \int_{4Q} |f| \right)^2. \end{aligned}$$

Further, by (1.1), for  $(y, t) \in T(2Q)$ ,

$$\begin{aligned} |(f\chi_{\mathbb{R}^n \setminus 4Q}) * \psi_t(y)| &\leq c_\psi t^\varepsilon \int_{\mathbb{R}^n \setminus 4Q} |f(\xi)| \frac{1}{(t + |y - \xi|)^{n+\varepsilon}} d\xi \\ &\leq c_{n,\psi} (t/\ell_Q)^\varepsilon \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int_{2^k Q} |f|. \end{aligned}$$

Hence, using Chebyshev's inequality and that  $\int_{\mathbb{R}^n} \Phi\left(\frac{x-y}{t\alpha}\right) dx \leq c_n(t\alpha)^n$ , we have

$$\begin{aligned} &(I_1(f\chi_{\mathbb{R}^n \setminus 4Q})\chi_Q)^*(\lambda|Q|/2) \\ &\leq \frac{2}{\lambda|Q|} \iint_{T(2Q)} \left( \int_{\mathbb{R}^n} \Phi\left(\frac{x-y}{t\alpha}\right) dx \right) |(f\chi_{\mathbb{R}^n \setminus 4Q}) * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \\ &\leq c_{n,\lambda,\psi} \alpha^n (1/\ell_Q)^{2\varepsilon} \left( \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2 \int_0^{2\ell_Q} t^{2\varepsilon-1} dt \\ &\leq c_{n,\lambda,\psi} \alpha^n \left( \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2. \end{aligned}$$

By Hölder's inequality,

$$\left( \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2 \leq \left( \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \right) \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2.$$

Combining this with the previous estimate and with (3.5) and (3.4) proves (3.3).

Let  $x, x_0 \in Q$ , and let us estimate now  $|I_2(f)(x) - I_2(f)(x_0)|$ . We have

$$\begin{aligned} &|I_2(f)(x) - I_2(f)(x_0)| \\ &\leq \sum_{k=1}^{\infty} \iint_{T(2^{k+1}Q) \setminus T(2^k Q)} \left| \Phi\left(\frac{x-y}{t\alpha}\right) - \Phi\left(\frac{x_0-y}{t\alpha}\right) \right| |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}}. \end{aligned}$$

Suppose  $(y, t) \in T(2^{k+1}Q) \setminus T(2^k Q)$ . If  $y \in 2^k Q$ , then  $t \geq 2^k \ell_Q$ . On the other hand, if  $y \in 2^{k+1}Q \setminus 2^k Q$ , then for any  $x \in Q$ ,  $|y - x| \geq$



$(2^k - 1/2)\ell_Q$ . Hence, if  $t < \frac{1}{2\alpha}(2^k - 1/2)\ell_Q$ , then  $|y - x|/\alpha t > 2$  and  $|y - x_0|/\alpha t > 2$ , and therefore,

$$\Phi\left(\frac{x-y}{t\alpha}\right) - \Phi\left(\frac{x_0-y}{t\alpha}\right) = 0.$$

Using also that

$$\left|\Phi\left(\frac{x-y}{t\alpha}\right) - \Phi\left(\frac{x_0-y}{t\alpha}\right)\right| \leq \frac{\sqrt{n}\ell_Q}{\alpha t} \|\nabla\Phi\|_{L^\infty},$$

we get

$$\begin{aligned} & \left|\Phi\left(\frac{x-y}{t\alpha}\right) - \Phi\left(\frac{x_0-y}{t\alpha}\right)\right| \chi_{\{T(2^{k+1}Q) \setminus T(2^kQ)\}}(y, t) \\ & \leq c_n \frac{\ell_Q}{\alpha t} \chi_{\{(y,t): y \in 2^{k+1}Q, 2^{k-2}\ell_Q/\alpha \leq t \leq 2^{k+1}\ell_Q\}}(y, t). \end{aligned}$$

Hence,

$$\begin{aligned} & \iint_{T(2^{k+1}Q) \setminus T(2^kQ)} \left|\Phi\left(\frac{x-y}{t\alpha}\right) - \Phi\left(\frac{x_0-y}{t\alpha}\right)\right| |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \\ & \leq c_n \frac{\ell_Q}{\alpha} \int_{2^{k-2}\ell_Q/\alpha}^{2^{k+1}\ell_Q} \int_{2^{k+1}Q} |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+2}} \leq c_n (J_1 + J_2), \end{aligned}$$

where

$$J_1 = \frac{\ell_Q}{\alpha} \int_{2^{k-2}\ell_Q/\alpha}^{2^{k+1}\ell_Q} \int_{2^{k+1}Q} |(f\chi_{2^{k+2}Q}) * \psi_t(y)|^2 \frac{dy dt}{t^{n+2}}$$

and

$$J_2 = \frac{\ell_Q}{\alpha} \int_{2^{k-2}\ell_Q/\alpha}^{2^{k+1}\ell_Q} \int_{2^{k+1}Q} |(f\chi_{\mathbb{R}^n \setminus 2^{k+2}Q}) * \psi_t(y)|^2 \frac{dy dt}{t^{n+2}}.$$

Let us first estimate  $J_1$ . Using Minkowski's integral inequality, we obtain

$$J_1 \leq \frac{\ell_Q}{\alpha} \left( \int_{2^{k+2}Q} |f(\xi)| \left( \int_{2^{k-2}\ell_Q/\alpha}^{2^{k+1}\ell_Q} \int_{2^{k+1}Q} \psi_t(y - \xi)^2 \frac{dy dt}{t^{n+2}} \right)^{1/2} d\xi \right)^2.$$

Since

$$\int_{2^{k+1}Q} \psi_t(y - \xi)^2 dy \leq \frac{\|\psi\|_{L^\infty}}{t^n} \|\psi_t\|_{L^1} = \frac{\|\psi\|_{L^\infty} \|\psi\|_{L^1}}{t^n},$$

we get

$$\begin{aligned} J_1 & \leq c_\psi \frac{\ell_Q}{\alpha} \left( \int_{2^{k+2}Q} |f(\xi)| d\xi \right)^2 \int_{2^{k-2}\ell_Q/\alpha}^\infty \frac{dt}{t^{2n+2}} \\ & \leq c_{n,\psi} \alpha^{2n} 2^{-k} \left( \frac{1}{|2^{k+2}Q|} \int_{2^{k+2}Q} |f(\xi)| d\xi \right)^2. \end{aligned}$$

We turn to the estimate of  $J_2$ . By (1.1), for  $(y, t) \in T(2^{k+1}Q)$ ,

$$\begin{aligned} |(f\chi_{\mathbb{R}^n \setminus 2^{k+2}Q}) * \psi_t(y)| &\leq c_\psi t^\varepsilon \int_{\mathbb{R}^n \setminus 2^{k+2}Q} |f(\xi)| \frac{1}{(t + |y - \xi|)^{n+\varepsilon}} d\xi \\ &\leq c_{n,\psi} (t/\ell_Q)^\varepsilon \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^i Q|} \int_{2^i Q} |f|. \end{aligned}$$

Therefore,

$$\begin{aligned} J_2 &\leq c_{n,\psi} \frac{\ell_Q}{\alpha} \left( \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^i Q|} \int_{2^i Q} |f| \right)^2 \frac{1}{\ell_Q^{2\varepsilon}} \int_{2^{k-2}\ell_Q/\alpha}^{2^{k+1}\ell_Q} \int_{2^{k+1}Q} \frac{dy dt}{t^{n+2-2\varepsilon}} \\ &\leq c_{n,\psi} \alpha^{n-2\varepsilon} 2^{(2\varepsilon-1)k} \left( \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^i Q|} \int_{2^i Q} |f| \right)^2. \end{aligned}$$

Combining the estimates for  $J_1$  and  $J_2$ , we obtain

$$\begin{aligned} |I_2(f)(x) - I_2(f)(x_0)| &\leq c_{n,\psi} \alpha^{2n} \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f(\xi)| d\xi \right)^2 \\ &\quad + c_{n,\psi} \alpha^{n-2\varepsilon} \sum_{k=1}^{\infty} \frac{2^{2\varepsilon k}}{2^k} \left( \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^i Q|} \int_{2^i Q} |f| \right)^2. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{2^{2\varepsilon k}}{2^k} \left( \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^i Q|} \int_{2^i Q} |f| \right)^2 \\ &\leq c_\varepsilon \sum_{k=1}^{\infty} \frac{2^{\varepsilon k}}{2^k} \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \left( \frac{1}{|2^i Q|} \int_{2^i Q} |f| \right)^2 \\ &\leq c_\varepsilon \sum_{k=1}^{\infty} \gamma(k, \varepsilon) \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2, \end{aligned}$$

where

$$\gamma(k, \varepsilon) = \begin{cases} \frac{1}{2^{\varepsilon k}}, & \varepsilon < 1 \\ \frac{k}{2^k}, & \varepsilon = 1. \end{cases}$$

Therefore,

$$|I_2(f)(x) - I_2(f)(x_0)| \leq c_{n,\psi} \alpha^{2n} \sum_{k=1}^{\infty} \gamma(k, \varepsilon) \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2.$$

From this and from (3.3),

$$\begin{aligned} \omega_\lambda(\tilde{S}_{\alpha,\psi}(f)^2; Q) &\leq (I_1(f)\chi_Q)^*(\lambda|Q|) + \|I_2(f) - I_2(f)(x_0)\|_{L^\infty(Q)} \\ &\leq c_{n,\lambda,\psi}\alpha^{2n} \sum_{k=0}^{\infty} \gamma(k, \varepsilon) \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2, \end{aligned}$$

which completes the proof.  $\square$

#### 4. PROOF OF THEOREM 1.1

**4.1. Several auxiliary operators.** Given a sparse family  $\mathcal{S} = \{Q_j^k\} \in \mathcal{D}$ , define

$$\mathcal{T}_{2,m}^{\mathcal{S}} f(x) = \left( \sum_{j,k} (f_{2^m Q_j^k})^2 \chi_{Q_j^k}(x) \right)^{1/2}.$$

In the case when  $m = 0$ , the following result was proved in [4].

**Lemma 4.1.** *For any  $1 < p < \infty$ ,*

$$\|\mathcal{T}_{2,0}^{\mathcal{S}}\|_{L^p(w)} \leq c_{n,p} [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})}.$$

Given a sparse family  $\mathcal{S} = \{Q_j^k\} \in \mathcal{D}$ , define

$$\mathcal{M}_m^{\mathcal{S}}(f, g)(x) = \sum_{j,k} (f_{2^m Q_{j,k}}) \left( \frac{1}{|2^m Q_j^k|} \int_{Q_j^k} g \right) \chi_{2^m Q_j^k}(x).$$

Applying Proposition 2.2, we decompose the cubes  $Q_j^k$  into  $2^n$  disjoint families  $F_i$  such that for any  $Q_j^k \in F_i$  there exists a cube  $P_{j,k}^{m,i} \in \mathcal{D}_i$  such that  $2^m Q_j^k \subset P_{j,k}^{m,i}$  and  $\ell_{P_{j,k}^{m,i}} \leq 6\ell_{2^m Q_j^k}$ . Hence,

$$(4.1) \quad \mathcal{M}_m^{\mathcal{S}}(f, g)(x) \leq 6^{2n} \sum_{i=1}^{2^n} \mathcal{M}_{i,m}^{\mathcal{S}}(f, g)(x),$$

where

$$\mathcal{M}_{i,m}^{\mathcal{S}}(f, g)(x) = \sum_{j,k} (f_{P_{j,k}^{m,i}}) \left( \frac{1}{|P_{j,k}^{m,i}|} \int_{Q_j^k} g \right) \chi_{P_{j,k}^{m,i}}(x).$$

The following statement was obtained in [5].

**Lemma 4.2.** *Suppose that the sum defining  $\mathcal{M}_{i,m}^{\mathcal{S}}(f, g)$  is finite. Then there are at most  $2^n$  cubes  $Q_\nu \in \mathcal{D}_i$  covering the support of  $\mathcal{M}_{i,m}^{\mathcal{S}}(f, g)$*

so that for any  $Q_\nu$  there are two sparse families  $\mathcal{S}_{i,1}$  and  $\mathcal{S}_{i,2}$  from  $\mathcal{D}_i$  such that for a.e.  $x \in Q_\nu$ ,

$$\mathcal{M}_{i,m}^{\mathcal{S}}(f, g)(x) \leq c_n m \sum_{\kappa=1}^2 \sum_{Q_j^k \in \mathcal{S}_{i,\kappa}} f_{Q_j^k} g_{Q_j^k} \chi_{Q_j^k}(x).$$

Observe that the proof of Lemma 4.2 is based on Theorem 2.3 along with [14, Lemma 3.2]. Formally Lemma 4.2 follows from [5, Lemma 4.2] taking there  $m = 2$  (which corresponds to a bilinear case) and  $l = m$ , and from the subsequent argument in [5, Section 4.2].

Let  $X$  be a Banach function space, and let  $X'$  denote the associate space (see [2, Ch. 1]). Given a Banach function space  $X$ , denote by  $X^{(2)}$  the space endowed with the norm

$$\|f\|_{X^{(2)}} = \||f|^2\|_X^{1/2}.$$

It is well known [16, Ch. 1] that  $X^{(2)}$  is also a Banach space.

**Lemma 4.3.** *For any Banach function space  $X$ ,*

$$\sup_{\mathcal{S} \in \mathcal{D}} \|\mathcal{T}_{2,m}^{\mathcal{S}} f\|_{X^{(2)}} \leq c_n m^{1/2} \max_{1 \leq i \leq 2^n} \sup_{\mathcal{S} \in \mathcal{D}_i} \|\mathcal{T}_{2,0}^{\mathcal{S}} f\|_{X^{(2)}}.$$

*Proof.* By the standard argument, one can assume that the sum defining  $\mathcal{T}_{2,m}^{\mathcal{S}} f$  is finite. Fix  $\mathcal{S} \in \mathcal{D}$ . By duality, there exists  $g \geq 0$  with  $\|g\|_{X'} = 1$  such that

$$\begin{aligned} (4.2) \quad \|\mathcal{T}_{2,m}^{\mathcal{S}} f\|_{X^{(2)}}^2 &= \int_{\mathbb{R}^n} (\mathcal{T}_{2,m}^{\mathcal{S}} f)^2 g \, dx = \sum_{j,k} (f_{2^m Q_j^k})^2 \int_{Q_j^k} g \\ &= \int_{\mathbb{R}^n} \mathcal{M}_m^{\mathcal{S}}(f, g) f \, dx. \end{aligned}$$

Observe that the sum defining  $\mathcal{M}_m^{\mathcal{S}}(f, g)$  is finite. By Lemma 4.2 and by Hölder's inequality,

$$\begin{aligned} \int_{Q_\nu} \mathcal{M}_{i,m}^{\mathcal{S}}(f, g) f \, dx &\leq c_n m \sum_{\kappa=1}^2 \sum_{Q_j^k \in \mathcal{S}_{i,\kappa}} (f_{Q_j^k})^2 \int_{Q_j^k} g \\ &\leq c_n m \sum_{\kappa=1}^2 \int_{\mathbb{R}^n} (\mathcal{T}_{2,0}^{\mathcal{S}_{i,\kappa}} f)^2 g \, dx \\ &\leq 2c_n m \sup_{\mathcal{S} \in \mathcal{D}_i} \|\mathcal{T}_{2,0}^{\mathcal{S}} f\|_{X^{(2)}}^2. \end{aligned}$$

Summing up over  $Q_\nu$  and using (4.1), we obtain

$$\int_{\mathbb{R}^n} \mathcal{M}_m^{\mathcal{S}}(f, g) f \, dx \leq c_n m \max_{1 \leq i \leq 2^n} \sup_{\mathcal{S} \in \mathcal{D}_i} \|\mathcal{T}_{2,0}^{\mathcal{S}} f\|_{X^{(2)}}^2.$$

This along with (4.2) completes the proof.  $\square$

**4.2. Proof of Theorem 1.1.** Let  $Q \in \mathcal{D}$ . Applying Theorem 2.3 along with Lemma 3.1, we get that there exists a sparse family  $\mathcal{S} = \{Q_j^k\} \in \mathcal{D}(Q)$  such that for a.e.  $x \in Q$ ,

$$|\tilde{S}_{\alpha,\psi}(f)(x)^2 - m_Q(\tilde{S}_{\alpha,\psi}(f)^2)| \leq c_{n,\psi} \alpha^{2n} \left( Mf(x)^2 + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} (\mathcal{T}_{2,m}^{\mathcal{S}} f(x))^2 \right).$$

Hence,

$$(4.3) \quad |\tilde{S}_{\alpha,\psi}(f)^2 - m_Q(\tilde{S}_{\alpha,\psi}(f)^2)|^{1/2} \leq c_{n,\psi} \alpha^n (Mf(x) + \mathcal{T}(f)(x)),$$

where

$$\mathcal{T}(f)(x) = \sum_{m=0}^{\infty} \frac{1}{2^{m\delta/2}} \mathcal{T}_{2,m}^{\mathcal{S}} f(x).$$

Assuming, for instance, that  $f \in L^1$ , and using (2.3) and (3.1), we get

$$\lim_{|Q| \rightarrow \infty} m_Q(\tilde{S}_{\alpha,\psi}(f)^2) = 0.$$

Therefore, letting  $Q$  to anyone of  $2^n$  quadrants and using Fatou's lemma, by (4.3) we obtain

$$(4.4) \quad \|\tilde{S}_{\alpha,\psi}(f)\|_{L^p(w)} \leq c_{n,\psi} \alpha^n (\|Mf\|_{L^p(w)} + \|\mathcal{T}(f)\|_{L^p(w)}).$$

Combining Lemma 4.1 and Lemma 4.3 with  $X = L^{3/2}(w)$  yields

$$\begin{aligned} \|\mathcal{T}(f)\|_{L^3(w)} &\leq \sum_{m=0}^{\infty} \frac{1}{2^{m\delta/2}} \|\mathcal{T}_{2,m}^{\mathcal{S}} f\|_{L^3(w)} \\ &\leq c_n \sum_{m=0}^{\infty} \frac{m^{1/2}}{2^{m\delta/2}} \max_{1 \leq i \leq 2^n} \sup_{\mathcal{S} \in \mathcal{D}_i} \|\mathcal{T}_{2,0}^{\mathcal{S}} f\|_{L^3(w)} \\ &\leq c_{n,\delta} [w]_{A_3}^{1/2} \|f\|_{L^3(w)}. \end{aligned}$$

Hence, by the sharp version of the Rubio de Francia extrapolation theorem (see [6] or [7]),

$$(4.5) \quad \|\mathcal{T}(f)\|_{L^p(w)} \leq c_{n,p,\delta} [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})} \|f\|_{L^p(w)} \quad (1 < p < \infty).$$

Thus, applying this result along with Buckley's estimate  $\|M\|_{L^p(w)} \leq c_{n,p} [w]_{A_p}^{\frac{1}{p-1}}$  (see [3]) and (4.4), we get

$$\|S_{\alpha,\psi}\|_{L^p(w)} \leq \|\tilde{S}_{\alpha,\psi}\|_{L^p(w)} \leq c_{n,p,\psi} \alpha^n [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})},$$

and therefore, the proof is complete.

## 5. CONCLUDING REMARKS

In a recent work [11], the following weak type estimate was obtained for  $G_\beta(f)$  (and hence for  $S_\psi(f)$ ): if  $1 < p < 3$ , then

$$\|G_\beta(f)\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p})} \Phi_p([w]_{A_p}) \|f\|_{L^p(w)},$$

where  $\Phi_p(t) = 1$  if  $1 < p < 2$  and  $\Phi_p(t) = 1 + \log t$  if  $p \geq 2$ . The proof was based on the local mean oscillation decomposition technique along with the estimate

$$(5.1) \quad \|\mathcal{T}_{2,0}^S f\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p})} \Phi_p([w]_{A_p}) \|f\|_{L^p(w)}.$$

Since the space  $L^{p,\infty}(w)$  is normable if  $p > 1$  (see, e.g., [2, p. 220]), combining Lemma 4.3 with  $X = L^{1+\varepsilon,\infty}(w)$ ,  $\varepsilon > 0$ , and (5.1) yields for  $2 < p < 3$  that

$$(5.2) \quad \|\mathcal{T}f\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p})} \Phi_p([w]_{A_p}) \|f\|_{L^p(w)}.$$

Hence, exactly as above, by (4.3) (and by the weak type estimate for  $M$  proved in [3]), we obtain

$$\|S_{\alpha,\psi}(f)\|_{L^{p,\infty}(w)} \lesssim \alpha^n [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p})} \Phi_p([w]_{A_p}) \|f\|_{L^p(w)} \quad (2 < p < 3).$$

We emphasize that our approach does not allow to extend this estimate to  $1 < p \leq 2$ . This is clearly related to the same problem with (5.2). The limitation  $2 < p < 3$  in (5.2) is due to Lemma 4.3 where the condition that  $X$  is a Banach function space was essential in the proof. This raises a natural question whether Lemma 4.3 holds under the condition that  $X$  is a quasi-Banach space. Observe that the same question can be asked regarding a recent estimate related  $X$ -norms of Calderón-Zygmund and dyadic positive operators [15].

## REFERENCES

- [1] P. Auscher, *Change of angle in tent spaces*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 5-6, 297-301.
- [2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York, 1988.
- [3] S.M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc., **340** (1993), no. 1, 253-272.
- [4] D. Cruz-Uribe, J.M. Martell and C. Pérez, *Sharp weighted estimates for classical operators*, Adv. Math., **229** (2012), no. 1, 408-441.
- [5] W. Damián, A.K. Lerner and C. Pérez, *Sharp weighted bounds for multilinear maximal functions and Calderón-Zygmund operators*, preprint. Available at <http://arxiv.org/abs/1211.5115>

- [6] O. Dragičević, L. Grafakos, M.C. Pereyra and S. Petermichl, *Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces*, Publ. Math., **49** (2005), no. 1, 73–91.
- [7] J. Duoandikoetxea, *Extrapolation of weights revisited: new proofs and sharp bounds*, J. Funct. Anal., **260** (2011), 1886–1901.
- [8] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.
- [9] J. García-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland (1985).
- [10] T. Hytönen and C. Pérez, *Sharp weighted bounds involving  $A_\infty$* , Analysis & PDE, to appear.
- [11] M.T. Lacey and J. Scurry, *Weighted weak type estimates for square functions*, preprint. Available at <http://arxiv.org/abs/1211.4219>
- [12] A.K. Lerner, *A pointwise estimate for the local sharp maximal function with applications to singular integrals*, Bull. London Math. Soc., **42** (2010), no. 5, 843–856.
- [13] A.K. Lerner, *Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals*, Adv. Math., **226** (2011), 3912–3926.
- [14] A.K. Lerner, *A simple proof of the  $A_2$  conjecture*, Int. Math. Res. Not. 2012, doi:10.1093/imrn/rns145.
- [15] A.K. Lerner, *On an estimate of Calderón-Zygmund operators by dyadic positive operators*, J. Anal. Math., to appear. Available at <http://arxiv.org/abs/1202.1860>
- [16] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, Berlin, 1979.
- [17] B. Muckenhoupt and R.L. Wheeden, *Norm inequalities for the Littlewood-Paley function  $g_\lambda^*$* , Trans. Amer. Math. Soc. **191** (1974), 95–111.
- [18] A. Torchinsky, *Real-variable methods in harmonic analysis*, Academic Press, 1986.
- [19] J.M. Wilson, *The intrinsic square function*, Rev. Mat. Iberoamericana, **23** (2007), 771–791.

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, 52900 RAMAT GAN,  
ISRAEL

*E-mail address:* `aklerner@netvision.net.il`